

# *Generalisation of the Yang-Mills Theory*

George Savvidy

*Institute of Nuclear and Particle Physics*

<sup>+</sup> *Demokritos National Research Center, Ag. Paraskevi, Athens, Greece*

## **Abstract**

We suggest an extension of the gauge principle which includes tensor gauge fields. In this extension of the Yang-Mills theory the vector gauge boson becomes a member of a bigger family of gauge bosons of arbitrary large integer spins. The proposed extension is essentially based on the extension of the Poincaré algebra and the existence of an appropriate transversal representations. The invariant Lagrangian is expressed in terms of new higher-rank field strength tensors. It does not contain higher derivatives of tensor gauge fields and all interactions take place through three- and four-particle exchanges with a dimensionless coupling constant. We calculated the scattering amplitudes of non-Abelian tensor gauge bosons at tree level, as well as their one-loop contribution into the Callan-Symanzik beta function. This contribution is negative and corresponds to the asymptotically free theory. Considering the contribution of tensorgluons of all spins into the beta function we found that it is leading to the theory which is conformally invariant at very high energies. The proposed extension may lead to a natural inclusion of the standard theory of fundamental forces into a larger theory in which vector gauge bosons, leptons and quarks represent a low-spin subgroup. We consider a possibility that inside the proton and, more generally, inside hadrons there are additional partons - tensorgluons, which can carry a part of the proton momentum. The extension of QCD influences the unification scale at which the coupling constants of the Standard Model merge, shifting its value to lower energies.

*Talk given at the "Conference on 60 Years of Yang-Mills Gauge Field Theories"*  
*Singapore 2015*

# 1 *Introduction*

It is well understood that the concept of local gauge invariance formulated by Yang and Mills [1] allows to define the non-Abelian gauge fields, to derive their dynamical field equations and to develop a universal point of view on matter interactions as resulting from the exchange of gauge quanta of different forms. The fundamental forces - electromagnetic, weak and strong interactions are successfully described by the non-Abelian Yang-Mills fields. The vector-like gauge particles - the photon,  $W^\pm$ ,  $Z$  and gluons mediate interaction between smallest constituents of matter - leptons and quarks.

The non-Abelian local gauge invariance, which was formulated by Yang and Mills [1], requires that all interactions must be invariant under independent rotations of internal charges at all space-time points. The gauge principle allows very little arbitrariness: the interaction of matter fields, which carry non-commuting internal charges, and the nonlinear self-interaction of gauge bosons are essentially fixed by the requirement of local gauge invariance, very similarly to the self-interaction of gravitons in general relativity [2, 3, 4, 5, 6, 7, 8].

It is therefore appealing to extend the gauge principle, which was elevated by Yang and Mills to a powerful constructive principle, so that it will define the interaction of matter fields which carry not only non-commutative internal charges, but also arbitrary large spins<sup>1</sup>. It seems that this will naturally lead to a theory in which fundamental forces will be mediated by integer-spin gauge quanta and that the Yang-Mills vector gauge boson will become a member of a bigger family of tensor gauge bosons [9, 10, 11, 50, 51]

The proposed extension of Yang-Mills theory is essentially based on the extension of the Poincaré algebra and the existence of an appropriate transversal representations of that algebra. The tensor gauge fields take value in extended Poincaré algebra. The invariant Lagrangian is expressed in terms of new higher-rank field strength tensors. The Lagrangian does not contain higher derivatives of tensor gauge fields and all interactions take place through three- and four-particle exchanges with a dimensionless coupling con-

---

<sup>1</sup>The research in high spin field theories has long and rich history. One should mention the early works of Majorana [23], Dirac [24], Fierz [25], Pauli [26], Schwinger [30], Singh and Hagen [31], Fronsdal [32], Weinberg [33], Minkowski [27], Brink et.al. [46], Berends, Burgers and Van Dam [41], Ginsburg and Tamm [34, 37], Nambu [38], Ramond [35], Brink [36], Fradkin [39], Vasiliev [40], Sagnotti, Sezgin and Sundel [42], Metsaev [43], Gabrielli [48], Castro [49], Manvelyan et.al. [44], and many other works (see also the references in [37, 40, 45]).

stant [9, 10, 11, 50, 51].

It is important to calculate the scattering amplitudes of non-Abelian tensor gauge bosons at tree level, as well as their one-loop contribution into the Callan-Symanzik beta function. This contribution is negative and corresponds to the asymptotically free theory. The proposed extension may lead to a natural inclusion of the standard theory of fundamental forces into a larger theory in which vector gauge bosons, leptons and quarks represent a low-spin subgroup [54, 55, 56].

In the line with the above development we considered a possible extension of QCD. In so extended QCD the spectrum of the theory contains new bosons, *the tensorgluons*, in addition to the quarks and gluons. The tensorgluons have zero electric charge, like gluons, but have a larger spin. Radiation of tensorgluons by gluons leads to a possible existence of tensorgluons inside the proton and, more generally, inside the hadrons. Due to the emission of tensorgluons part of the proton momentum which is carried by the neutral constituents can be shared between gluons and tensorgluons. The density of neutral partons is therefore given by the sum:  $G(x, t) + T(x, t)$ , where  $T(x, t)$  is the density of the tensorgluons [54, 55, 56]. To disentangle these contributions and to decide which piece of the neutral partons is the contribution of gluons  $G(x, t)$  and which one is of the tensorgluons one should measure the helicities of the neutral components, which seems to be a difficult task.

The extension of QCD influences the unification scale at which the coupling constants of the Standard Model merge. We observed that the unification scale at which standard coupling constants are merging is shifted to lower energies telling us that it may be that a new physics is round the corner. Whether all these phenomena are consistent with experiment is an open question.

The paper is organised as follows. In Section 2 we shall define the composite gauge field  $\mathcal{A}_\mu(x, e)$ , which depends on the space-time coordinates  $x_\mu$  and the new space-like vector variable  $e^\lambda$ . The high-rank tensor gauge fields  $A_{\mu\lambda_1\dots\lambda_s}^a(x)$  appear in the expansion of  $\mathcal{A}_\mu(x, e)$  over the vector variable. We introduce a corresponding extension of the Poincaré algebra  $L_G(\mathcal{P})$  and consider the high-rank fields as tensor gauge fields taking value in algebra  $L_G(\mathcal{P})$ . In Section 3 we shall describe the transversal representation  $L^\perp$  of the generators of the algebra  $L_G(\mathcal{P})$ , their helicity content and their invariant scalar products. The fact that the representation of the generators is transversal plays an important role

in the definition of the gauge field  $\mathcal{A}_\mu(x, e)$ . In transversal representation the tensor gauge fields are projecting out into the plane transversal to the momentum and contain only positive space-like components of a definite helicity.

In Section 4 we shall define the gauge transformation of the gauge fields, the field strength tensors and the invariant Lagrangian. The kinetic term describes the propagation of positive definite helicity states. The helicity spectrum of the propagating modes is consistent with the helicity spectrum which appears in the projection of the tensor gauge fields into transversal generators  $L^\perp$ . The Lagrangian defines not only a free propagation of tensor gauge bosons, but also their interactions. The interaction diagrams for the lower-rank bosons are presented on Fig.1-2. The high-rank bosons interact through the triple and quartic interaction vertices with a dimensionless coupling constant. In Section 5 we shall calculate and study the scattering amplitudes of the vector and tensor gauge bosons and their splitting amplitudes by using spinor representation of the momenta and polarisation tensors.

In Section 6 we shall consider a possibility that inside the proton and, more generally, inside hadrons there are additional partons - tensorgluons, which can carry a part of the proton momentum. We generalise the DGLAP equation which includes the splitting probabilities of the gluons into tensorgluons and calculated the one-loop Callan-Simanzik beta function. This contribution is negative and corresponds to the asymptotically free theory. Considering the contribution of tensorgluons of all spins into the beta function we found that it is leading to the theory which is *conformally invariant* at very high energies. In Section 7 we observed that the unification scale at which standard coupling constants are merging is shifted to lower energies. In conclusion we summarise the results and discuss the challenges of the experimental verification of the suggested model.

## 2 *Tensor Gauge Fields and Extended Poincaré Algebra*

The gauge fields are defined as rank- $(s + 1)$  tensors [9, 10, 11]

$$A_{\mu\lambda_1\dots\lambda_s}^a(x), \quad s = 0, 1, 2, \dots \quad (2.1)$$

and are totally symmetric with respect to the indices  $\lambda_1\dots\lambda_s$ . A priori the tensor fields have no symmetries with respect to the first index  $\mu$ . The index  $a$  numerates the generators

$L_a$  of the Lie algebra  $L_G$  of a compact Lie group  $G$  with totally antisymmetric structure constants  $f_{abc}$ .

The tensor fields (2.1) can be considered as the components of a composite gauge field  $\mathcal{A}_\mu(x, e)$  which depends on additional translationally invariant space-like unit vector [11, 15, 16, 17]:

$$e_\lambda e^\lambda = -1. \quad (2.2)$$

A similar vector variable, in addition to the space-time coordinate  $x$ , was introduced earlier by Yakawa [12], Fierz [13], Wigner [14], Ginzburg and Tamm [34, 37] and others [40]. The variable  $e^\lambda$  is also reminiscent to the Grassmann variable  $\theta$  in supersymmetric theories where the superfield  $\Psi(x, \theta)$  depends on two variables  $x$  and  $\theta$  [18, 19]. We shall consider all tensor gauge fields (2.1) as the components appearing in the expansion over the above mentioned vector variable [11]:

$$\mathcal{A}_\mu(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} A_{\mu\lambda_1\ldots\lambda_s}^a(x) L_a e^{\lambda_1} \ldots e^{\lambda_s}. \quad (2.3)$$

The gauge field  $A_{\mu\lambda_1\ldots\lambda_s}^a$  carries indices  $a, \lambda_1, \ldots, \lambda_s$  which are labelling the generators  $L_a^{\lambda_1\ldots\lambda_s} = L_a e^{\lambda_1} \ldots e^{\lambda_s}$  of extended current algebra  $L_G$  associated with the Lie algebra  $L_G$  [11, 50]. The algebra  $L_G$  has infinitely many generators  $L_a^{\lambda_1\ldots\lambda_s}$  and is given by the commutator [11, 50, 51]

$$[L_a^{\lambda_1\ldots\lambda_k}, L_b^{\lambda_{k+1}\ldots\lambda_s}] = i f_{abc} L_c^{\lambda_1\ldots\lambda_s}, \quad s = 0, 1, 2, \ldots \quad (2.4)$$

The generators  $L_a^{\lambda_1\ldots\lambda_s}$  commute to themselves forming an infinite series of commutators of current algebra  $L_G$  which cannot be truncated, so that the index  $s$  runs from zero to infinity. Because the generators  $L_a^{\lambda_1\ldots\lambda_s}$  are space-time tensors, the full algebra should include the Poincaré generators  $P^\mu, M^{\mu\nu}$  as well. This naturally leads to the extension  $L_G(\mathcal{P})$  of the Poincaré algebra  $L_{\mathcal{P}}$  [50, 51, 52]:

$$\begin{aligned} [P^\mu, P^\nu] &= 0, \\ [M^{\mu\nu}, P^\lambda] &= \eta^{\nu\lambda} P^\mu - \eta^{\mu\lambda} P^\nu, \\ [M^{\mu\nu}, M^{\lambda\rho}] &= \eta^{\mu\rho} M^{\nu\lambda} - \eta^{\mu\lambda} M^{\nu\rho} + \eta^{\nu\lambda} M^{\mu\rho} - \eta^{\nu\rho} M^{\mu\lambda}, \\ [P^\mu, L_a^{\lambda_1\ldots\lambda_s}] &= 0, \\ [M^{\mu\nu}, L_a^{\lambda_1\ldots\lambda_s}] &= \eta^{\nu\lambda_1} L_a^{\mu\lambda_2\ldots\lambda_s} - \eta^{\mu\lambda_1} L_a^{\nu\lambda_2\ldots\lambda_s} + \ldots + \eta^{\nu\lambda_s} L_a^{\mu\lambda_1\ldots\lambda_{s-1}} - \eta^{\mu\lambda_s} L_a^{\nu\lambda_1\ldots\lambda_{s-1}}, \\ [L_a^{\lambda_1\ldots\lambda_k}, L_b^{\lambda_{k+1}\ldots\lambda_s}] &= i f_{abc} L_c^{\lambda_1\ldots\lambda_s}. \end{aligned} \quad (2.5)$$

We have here an extension of the Poincaré algebra by generators  $L_a^{\lambda_1 \dots \lambda_s}$  which carry the *internal charges and spins*. The algebra  $L_G(\mathcal{P})$  incorporates the Poincaré algebra  $L_{\mathcal{P}}$  and an internal algebra  $L_G$  in a nontrivial way, which is different from the direct product.

There is no conflict with the Coleman-Mandula theorem [20, 21] because the theorem applies to the symmetries that act on S-matrix elements and not on all the other symmetries that occur in quantum field theory. The above symmetry group (2.5) is the symmetry which acts on the gauge field  $\mathcal{A}_\mu(x, e)$  and is not the symmetry of the S-matrix. The theorem assumes among other things that the vacuum is nondegenerate and that there are no massless particles in the spectrum. As we shall see, the spectrum of the extended Yang-Mills theory is massless.

*In order to define the gauge field  $\mathcal{A}_\mu(x, e)$  in (2.3) and find out its helicity content one should specify the representation of the generators  $L_a^{\lambda_1 \dots \lambda_s}$  in algebra (2.5). In the next section we shall describe the so called transversal representation, which is used to define the tensor gauge fields in the decomposition (2.3).*

### 3 Transversal Representation of Algebra $L_G(\mathcal{P})$

The important property of the algebra (2.5) is its invariance with respect to the following "gauge" transformations [50, 51, 52]:

$$\begin{aligned} L_a^{\lambda_1 \dots \lambda_s} &\rightarrow L_a^{\lambda_1 \dots \lambda_s} + \sum_1 P^{\lambda_1} L_a^{\lambda_2 \dots \lambda_s} + \sum_2 P^{\lambda_1} P^{\lambda_2} L_a^{\lambda_3 \dots \lambda_s} + \dots + P^{\lambda_1} \dots P^{\lambda_s} L_a \\ M^{\mu\nu} &\rightarrow M^{\mu\nu}, \quad P^\lambda \rightarrow P^\lambda, \end{aligned} \quad (3.1)$$

where the sums  $\sum_1, \sum_2, \dots$  are over all inequivalent index permutations. The above transformations contain polynomials of the momentum operator  $P^\lambda$  and are reminiscent of the gauge field transformations. This is "off-shell" symmetry because the invariant operator  $P^2$  can have any value. As a result, to any given representation of  $L_a^{\lambda_1 \dots \lambda_s}$ ,  $s = 1, 2, \dots$  one can add the longitudinal terms, as it follows from the transformation (3.1). All representations are therefore defined modulo longitudinal terms, and we can identify these generators as "gauge generators".

The second general property of the extended algebra is that each gauge generator  $L_a^{\lambda_1 \dots \lambda_s}$  cannot be realised as an irreducible representation of the Poincaré algebra of a definite helicity, i.e. to be a *symmetric and traceless tensor*. The reason is that the commutator of two symmetric traceless generators in (2.5) is not any more a traceless

tensor. Therefore the generators  $L_a^{\lambda_1 \dots \lambda_s}$  realise a reducible representation of the Poincaré algebra and each of them carries a spectrum of helicities, which we shall describe below.

The algebra  $L_G(\mathcal{P})$  has representation in terms of differential operators of the following general form:

$$\begin{aligned} P^\mu &= k^\mu, \\ M^{\mu\nu} &= i(k^\mu \frac{\partial}{\partial k_\nu} - k^\nu \frac{\partial}{\partial k_\mu}) + i(e^\mu \frac{\partial}{\partial e_\nu} - e^\nu \frac{\partial}{\partial e_\mu}), \\ L_a^{\lambda_1 \dots \lambda_s} &= e^{\lambda_1} \dots e^{\lambda_s} \otimes L_a, \end{aligned} \quad (3.2)$$

where  $e^\lambda$  is a translationally invariant space-like unite vector (2.2). The vector space of a representation is parameterised by the momentum  $k^\mu$  and translationally invariant vector variables  $e^\lambda$ :

$$\Psi(k^\mu, e^\lambda). \quad (3.3)$$

The irreducible representations can be obtained from (3.2) by imposing invariant constraints on the vector space of functions (3.3) of the following form [14, 22, 12, 13]:

$$k^2 = 0, \quad k^\mu e_\mu = 0, \quad e^2 = -1. \quad (3.4)$$

These equations have a unique solution [14]

$$e^\mu = \chi k^\mu + e_1^\mu \cos \varphi + e_2^\mu \sin \varphi, \quad (3.5)$$

where  $e_1^\mu = (0, 1, 0, 0)$ ,  $e_2^\mu = (0, 0, 1, 0)$  when  $k^\mu = \omega(1, 0, 0, 1)$ . The  $\chi$  and  $\varphi$  remain as independent variables on the cylinder  $\varphi \in S^1, \chi \in R^1$ . The invariant subspace of functions (3.3) now reduces to the following form:

$$\Psi(k^\mu, e^\nu) \delta(k^2) \delta(k \cdot e) \delta(e^2 + 1) = \Phi(k^\mu, \varphi, \chi). \quad (3.6)$$

If we take into account (3.5) the generators  $L_a^{\lambda_1 \dots \lambda_s} = e^{\lambda_1} \dots e^{\lambda_s} \otimes L_a$ , it takes the following form:

$$L_a^{\perp \lambda_1 \dots \lambda_s} = \prod_{n=1}^s (\chi k^{\lambda_n} + e_1^{\lambda_n} \cos \varphi + e_2^{\lambda_n} \sin \varphi) \otimes L_a. \quad (3.7)$$

This is a purely transversal representation because of (3.4):

$$k_{\lambda_1} L_a^{\perp \lambda_1 \dots \lambda_s} = 0, \quad s = 1, 2, \dots \quad (3.8)$$

The generators  $L_a^{\perp \lambda_1 \dots \lambda_s}$  carry helicities in the following range:

$$h = (s, s-2, \dots, -s+2, -s), \quad (3.9)$$

in total  $s+1$  states. Indeed, this can be deduced from the explicit representation (3.7) by using helicity polarisation vectors  $e_{\pm}^{\lambda} = (e_1^{\lambda} \mp i e_2^{\lambda})/2$ :

$$L_a^{\perp \lambda_1 \dots \lambda_s} = \prod_{n=1}^s (\chi k^{\lambda_n} + e^{i\varphi} e_+^{\lambda_n} + e^{-i\varphi} e_-^{\lambda_n}) \oplus L_a. \quad (3.10)$$

Performing the multiplication in (3.10) and collecting the terms of a given power of momentum we shall get the following expression:

$$\begin{aligned} L_a^{\perp \mu_1 \dots \mu_s} &= \prod_{n=1}^s (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n}) \oplus L_a + \\ &+ \sum_1 \chi k^{\lambda_1} \prod_{n=1}^{s-1} (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n}) \oplus L_a + \dots + \chi k^{\lambda_1} \dots \chi k^{\lambda_s} \oplus L_a, \end{aligned} \quad (3.11)$$

where the first term  $\prod_{n=1}^s (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n})$  represents the *helicity generators*  $(L_a^{+\dots+}, \dots, L_a^{-\dots-})$ , while their helicity spectrum is described by the formula (3.9). The rest of the terms are purely longitudinal and proportional to the increasing powers of momentum  $k$ . The last formula also illustrates the realisation of the transformation (3.1), that is, the helicity generators  $(L_a^{+\dots+}, \dots, L_a^{-\dots-})$  are defined modulo longitudinal terms proportional to  $k^{\lambda_1} \dots k^{\lambda_n}, n = 1, \dots, s$ .

The very fact that the representation of the generators  $L_a^{\perp \lambda_1 \dots \lambda_s}$  is transversal plays an important role in the definition of the gauge field  $\mathcal{A}_{\mu}(x, e)$  in (2.3). Indeed, substituting the transversal representation (3.11) of the generators  $L_a^{\perp \lambda_1 \dots \lambda_s}$  into the expansion (2.3) and collecting the terms in front of the helicity generators  $(L_a^{+\dots+}, \dots, L_a^{-\dots-})$  we shall get

$$\begin{aligned} \mathcal{A}_{\mu}(x, e) &= \sum_{s=0}^{\infty} \frac{1}{s!} (\tilde{A}_{\mu\lambda_1 \dots \lambda_s}^a e_+^{\lambda_1} \dots e_+^{\lambda_s} \oplus L_a + \dots + \tilde{A}_{\mu\lambda_1 \dots \lambda_s}^a e_-^{\lambda_1} \dots e_-^{\lambda_s} \oplus L_a) \\ &= \sum_{s=0}^{\infty} \frac{1}{s!} (\tilde{A}_{\mu+\dots+}^a L_a^{+\dots+} + \dots + \tilde{A}_{\mu-\dots-}^a L_a^{-\dots-}), \end{aligned} \quad (3.12)$$

where  $s$  is the number of negative indices. This formula represents the projection  $\tilde{A}_{\mu\lambda_1 \dots \lambda_s}^a$  of the components of the non-Abelian tensor gauge field  $A_{\mu\lambda_1 \dots \lambda_s}^a$  into the plane transversal to the momentum. The projection contains only positive definite space-like components



of the helicities [50, 51, 52]:

$$h = \pm(s+1), \quad \frac{\pm(s-1)}{\pm(s-1)}, \quad \frac{\pm(s-3)}{\pm(s-3)}, \quad \dots, \quad (3.13)$$

where the lower helicity states have double degeneracy. The analysis of the kinetic terms of the Lagrangian and of the corresponding equation of motions, which will be considered in the next section, confirms that indeed the propagating degrees of freedom are described by helicities (3.13).

In order to define the gauge invariant Lagrangian one should know the Killing metric of the algebra  $L_G(\mathcal{P})$ . The explicit transversal representation of the  $L_G(\mathcal{P})$  generators given above (3.7), (3.10) and (3.11) allows to calculate the corresponding Killing metric [50, 51, 53]:

$$L_G : \quad \langle L_a; L_b \rangle = \delta_{ab}, \quad (3.14)$$

$$\begin{aligned} L_{\mathcal{P}} : \quad \langle P^\mu; P^\nu \rangle &= 0 \\ \langle M_{\mu\nu}; P_\lambda \rangle &= 0 \\ \langle M^{\mu\nu}; M^{\lambda\rho} \rangle &= \eta^{\mu\lambda}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\lambda} \end{aligned} \quad (3.15)$$

$$\begin{aligned} L_G(\mathcal{P}) : \quad \langle P^\mu; L_a^\perp{}^{\lambda_1 \dots \lambda_s} \rangle &= 0, \\ \langle M^{\mu\nu}; L_a^\perp{}^{\lambda_1 \dots \lambda_s} \rangle &= 0, \end{aligned} \quad (3.16)$$

$$\begin{aligned} \langle L_a; L_b^\perp{}^{\lambda_1} \rangle &= 0, \\ \langle L_a^\perp{}^{\lambda_1}; L_b^\perp{}^{\lambda_2} \rangle &= \delta_{ab} \bar{\eta}^{\lambda_1 \lambda_2}, \\ \langle L_a; L_b^\perp{}^{\lambda_1 \lambda_2} \rangle &= \delta_{ab} \bar{\eta}^{\lambda_1 \lambda_2}, \\ \langle L_a^\perp{}^{\lambda_1}; L_b^\perp{}^{\lambda_2 \lambda_3} \rangle &= 0, \\ &\dots\dots\dots \\ \langle L_a^\perp{}^{\lambda_1 \dots \lambda_n}; L_b^\perp{}^{\lambda_{n+1} \dots \lambda_{2s+1}} \rangle &= 0, \quad s = 0, 1, 2, 3, \dots \\ \langle L_a^\perp{}^{\lambda_1 \dots \lambda_n}; L_b^\perp{}^{\lambda_{n+1} \dots \lambda_{2s}} \rangle &= \delta_{ab} s! (\bar{\eta}^{\lambda_1 \lambda_2} \bar{\eta}^{\lambda_3 \lambda_4} \dots \bar{\eta}^{\lambda_{2s-1} \lambda_{2s}} + \text{perm}), \end{aligned} \quad (3.17)$$

where  $\bar{\eta}^{\lambda_1 \lambda_2}$  is the projector into the two-dimensional plane transversal to the momentum  $k^\mu$  [30]:

$$\bar{\eta}^{\lambda_1 \lambda_2} = \frac{k^{\lambda_1} \bar{k}^{\lambda_2} + \bar{k}^{\lambda_1} k^{\lambda_2}}{k \bar{k}} - \eta^{\lambda_1 \lambda_2}, \quad k_{\lambda_1} \bar{\eta}^{\lambda_1 \lambda_2} = k_{\lambda_2} \bar{\eta}^{\lambda_1 \lambda_2} = 0, \quad (3.18)$$

and  $\bar{k}^\mu = \omega(1, 0, 0, -1)$ . It follows then that the transversality conditions (3.8) are fulfilled:

$$k_{\lambda_i} \langle L_a^\perp{}^{\lambda_1 \dots \lambda_n}; L_b^\perp{}^{\lambda_{n+1} \dots \lambda_{2s}} \rangle = 0, \quad i = 1, 2, \dots, 2s. \quad (3.19)$$

The Killing metric on the internal  $L_G$  and on the Poincaré  $L_{\mathcal{P}}$  subalgebras (3.14), (3.15) are well known. The important conclusion which follows from the above result is that the Poincaré generators  $P^\mu, M^{\mu\nu}$  are orthogonal to the gauge generators  $L_a^{\lambda_1 \dots \lambda_s}$  (3.16). The last formulas (3.17) represent the Killing metric on the  $L_{\mathcal{G}}$  current algebra (2.4), (2.5) and will be used in the definition of the Lagrangian in the next section. It should be stressed that the metric (3.17) is defined modulo longitudinal terms. This is because under the "gauge" transformation of the generators (3.1) the metric will receive terms which are polynomial in momentum. The provided metric (3.17) is written in a particular gauge. This peculiar property of the metric is mirrored in the definition of the Lagrangian which can be written in different gauges. The spectrum of the propagating modes does not depend on the gauges chosen, as one can get convinced by inspecting the expression (3.12).

Notice that the reducible representation (3.2), without any of the constraints (3.4), should also be considered, as well as the representation in which only the last constrain in (3.4) is imposed. In that cases the transversality of the representation (3.19) will be lost, but instead one arrives to the homogeneous Killing metric in (3.17)  $\bar{\eta}^{\lambda_1 \lambda_2} \rightarrow \eta^{\lambda_1 \lambda_2}$  and the longitudinal terms which can be gauged away.

*With this Killing metric in hands one can define the Lagrangian of the theory.*

## 4 The Lagrangian

The gauge transformation of the field  $\mathcal{A}_\mu(x, e)$  is defined as [1, 11, 50]

$$\mathcal{A}'_\mu(x, e) = U(\xi) \mathcal{A}_\mu(x, e) U^{-1}(\xi) - \frac{i}{g} \partial_\mu U(\xi) U^{-1}(\xi), \quad (4.1)$$

where the group parameter  $\xi(x, e)$

$$U(\xi) = e^{i\xi(x, e)}$$

has the decomposition [11, 50]

$$\xi(x, e) = \sum_s \frac{1}{s!} \xi_{\lambda_1 \dots \lambda_s}^a(x) L_a e^{\lambda_1} \dots e^{\lambda_s}$$

and  $\xi_{\lambda_1 \dots \lambda_s}^a(x)$  are totally symmetric gauge parameters. Using the commutator of the covariant derivatives  $\nabla_\mu^{ab} = (\partial_\mu - ig\mathcal{A}_\mu(x, e))^{ab}$

$$[\nabla_\mu, \nabla_\nu]^{ab} = gf^{acb}\mathcal{G}_{\mu\nu}^c, \quad (4.2)$$

we can define the extended field strength tensor

$$\mathcal{G}_{\mu\nu}(x, e) = \partial_\mu \mathcal{A}_\nu(x, e) - \partial_\nu \mathcal{A}_\mu(x, e) - ig[\mathcal{A}_\mu(x, e) \mathcal{A}_\nu(x, e)], \quad (4.3)$$

which transforms homogeneously:

$$\mathcal{G}'_{\mu\nu}(x, e) = U(\xi)\mathcal{G}_{\mu\nu}(x, e)U^{-1}(\xi). \quad (4.4)$$

It is useful to have an explicit expression for the transformation law of the field components [9, 10, 11]:

$$\begin{aligned} \delta A_\mu^a &= (\delta^{ab}\partial_\mu + gf^{acb}A_\mu^c)\xi^b, \\ \delta A_{\mu\nu}^a &= (\delta^{ab}\partial_\mu + gf^{acb}A_\mu^c)\xi_\nu^b + gf^{acb}A_{\mu\nu}^c\xi^b, \\ \delta A_{\mu\nu\lambda}^a &= (\delta^{ab}\partial_\mu + gf^{acb}A_\mu^c)\xi_{\nu\lambda}^b + gf^{acb}(A_{\mu\nu}^c\xi_\lambda^b + A_{\mu\lambda}^c\xi_\nu^b + A_{\nu\lambda}^c\xi_\mu^b), \\ \dots\dots\dots &\dots\dots\dots \end{aligned} \quad (4.5)$$

These extended gauge transformations generate a closed algebraic structure. The component field strengths tensors take the following form [9, 10, 11]:

$$\begin{aligned} G_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \\ G_{\mu\nu,\lambda}^a &= \partial_\mu A_{\nu\lambda}^a - \partial_\nu A_{\mu\lambda}^a + gf^{abc}(A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c), \\ G_{\mu\nu,\lambda\rho}^a &= \partial_\mu A_{\nu\lambda\rho}^a - \partial_\nu A_{\mu\lambda\rho}^a + gf^{abc}(A_\mu^b A_{\nu\lambda\rho}^c + A_{\mu\lambda}^b A_{\nu\rho}^c + A_{\mu\rho}^b A_{\nu\lambda}^c + A_{\nu\lambda}^b A_{\mu\rho}^c), \\ \dots\dots\dots &\dots\dots\dots \end{aligned} \quad (4.6)$$

and transform homogeneously with respect to the transformations (4.5):

$$\begin{aligned} \delta G_{\mu\nu}^a &= gf^{abc}G_{\mu\nu}^b\xi^c, \\ \delta G_{\mu\nu,\lambda}^a &= gf^{abc}(G_{\mu\nu,\lambda}^b\xi^c + G_{\mu\nu}^b\xi_\lambda^c), \\ \delta G_{\mu\nu,\lambda\rho}^a &= gf^{abc}(G_{\mu\nu,\lambda\rho}^b\xi^c + G_{\mu\nu,\lambda}^b\xi_\rho^c + G_{\mu\nu,\rho}^b\xi_\lambda^c + G_{\mu\nu}^b\xi_{\lambda\rho}^c), \\ \dots\dots\dots &\dots\dots\dots \end{aligned} \quad (4.7)$$

The field strength tensors are antisymmetric in their first two indices and are totally symmetric with respect to the rest of the indices. The symmetry properties of the field strength  $G_{\mu\nu,\lambda_1\ldots\lambda_s}^a$  remain invariant in the course of these transformations.

The first gauge invariant density is given by the expression [9, 10, 11]

$$\mathcal{L}(x) = \langle \mathcal{L}(x, e) \rangle = -\frac{1}{4} \langle \mathcal{G}_{\mu\nu}^a(x, e) \mathcal{G}^{a\mu\nu}(x, e) \rangle, \quad (4.8)$$

where the trace of the generators is given in (3.17). One can get convinced that the variation of the (4.8) with respect to the gauge transformations (4.1) and (4.4) vanishes:

$$\delta \mathcal{L}(x, e) = -\frac{1}{2} \mathcal{G}_{\mu\nu}^a(x, e) g f^{abc} \mathcal{G}^{b\mu\nu}(x, e) \xi^c(x, e) = 0.$$

The invariant density (4.8) allows to extract *gauge invariant, totally symmetric, tensor densities*  $\mathcal{L}_{\lambda_1\ldots\lambda_s}(x)$  by using expansion with respect to the vector variable  $e^\lambda$ :

$$\mathcal{L}(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} \mathcal{L}_{\lambda_1\ldots\lambda_s}(x) e^{\lambda_1} \ldots e^{\lambda_s}. \quad (4.9)$$

In particular, the expansion term which is quadratic in powers of  $e^\lambda$  is

$$\mathcal{L}_{\lambda_1\lambda_2} = -\frac{1}{4} G_{\mu\nu,\lambda_1}^a G_{\mu\nu,\lambda_2}^a - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu,\lambda_1\lambda_2}^a. \quad (4.10)$$

The gauge invariant density thus can be represented in the following form [9, 10, 11]:

$$\mathcal{L}(x) = \langle \mathcal{L}(x, e) \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} \mathcal{L}_{\lambda_1\ldots\lambda_s}(x) \langle e^{\lambda_1} \ldots e^{\lambda_s} \rangle \quad (4.11)$$

and the density for the lower-rank tensor fields is

$$\mathcal{L}_2 = -\frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda}^a - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a.$$

Let us consider the second gauge invariant density of the form [9, 10, 11]

$$\mathcal{L}'(x) = \langle \mathcal{L}'(x, e) \rangle = \frac{1}{4} \langle \mathcal{G}_{\mu\rho_1}^a(x, e) e^{\rho_1} \mathcal{G}^{a\mu}{}_{\rho_2}(x, e) e^{\rho_2} \rangle'. \quad (4.12)$$

It is gauge invariant because its variation is also equal to zero:

$$\begin{aligned} \delta \mathcal{L}'(x, e) &= \frac{1}{4} g f^{acb} \mathcal{G}_{\mu\rho_1}^c(x, e) e^{\rho_1} \xi^b(x, e) \mathcal{G}^{a\mu}{}_{\rho_2}(x, e) e^{\rho_2} + \\ &+ \frac{1}{4} \mathcal{G}_{\mu\rho_1}^a(x, e) e^{\rho_1} g f^{acb} \mathcal{G}^{c\mu}{}_{\rho_2}(x, e) e^{\rho_2} \xi^b(x, e) = 0. \end{aligned} \quad (4.13)$$

The Lagrangian density (4.12) generates the second series of *gauge invariant tensor densities*  $(\mathcal{L}'_{\rho_1\rho_2})_{\lambda_1\dots\lambda_s}(x)$  when we expand it in powers of the vector variable  $e^\lambda$ :

$$\mathcal{L}'(x) = \langle \mathcal{L}'(x, e) \rangle = \sum_{s=0}^{\infty} \frac{1}{s!} (\mathcal{L}'_{\rho_1\rho_2})_{\lambda_1\dots\lambda_s}(x) \langle e^{\rho_1} e^{\rho_2} e^{\lambda_1} \dots e^{\lambda_s} \rangle'. \quad (4.14)$$

The term quartic in variable  $e^\lambda$  after contraction of the vector variables takes the following form:

$$\mathcal{L}'_2 = \frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^a + \frac{1}{4} G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda}^a + \frac{1}{2} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a. \quad (4.15)$$

One can get convinced that it is gauge invariant under the transformation (4.5) and (4.7). The total Lagrangian density is a sum of two invariants (4.8) and (4.12):

$$L = \mathcal{L} + \mathcal{L}' = -\frac{1}{4} \langle \mathcal{G}_{\mu\nu}^a(x, e) \mathcal{G}^{a\mu\nu}(x, e) \rangle + \frac{1}{4} \langle \mathcal{G}_{\mu\rho_1}^a(x, e) e^{\rho_1} \mathcal{G}^{a\mu}{}_{\rho_2}(x, e) e^{\rho_2} \rangle'. \quad (4.16)$$

The Lagrangian for the lower-rank tensor gauge fields has the following form:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}'_2 + \dots = & -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a \\ & -\frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda}^a - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a \\ & + \frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^a + \frac{1}{4} G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda}^a + \frac{1}{2} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a + \dots \end{aligned} \quad (4.17)$$

The above Lagrangian defines the kinetic operators for the rank-1  $A_\mu^a$  and rank-2  $A_{\mu\lambda_1}^a$  fields, as well as trilinear and quartic interactions with the *dimensionless coupling constant*  $g$  (see Fig.1-2).

As we found in [9, 10, 11], the corresponding free field equations coincide with the equations introduced in the classical works [25, 26, 27] and describe the propagation of the *helicity-two and zero*  $h = \pm 2, 0$  *massless charged tensor gauge bosons*, and there are no propagating negative norm states. This is in agreement with the spectrum presented in (3.13). The next term in expansion of the Lagrangian density has the following form [28, 29]:

$$\begin{aligned} \mathcal{L}_3 + \mathcal{L}'_3 = & -\frac{1}{4} G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - \frac{1}{8} G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - \frac{1}{2} G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - \frac{1}{8} G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a + \\ & + \frac{1}{3} G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + \frac{1}{3} G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + \frac{1}{3} G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a + \\ & + \frac{1}{3} G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a + \frac{2}{3} G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a + \frac{1}{3} G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a + \frac{1}{3} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a \end{aligned} \quad (4.18)$$

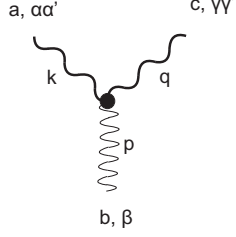


Figure 1: The interaction vertex for the vector gauge boson  $V$  and two tensor gauge bosons  $T$  - the VTT vertex -  $\mathcal{V}_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}^{abc}(k, p, q)$  in non-Abelian tensor gauge field theory [11]. Vector gauge bosons are conventionally drawn as thin wave lines, tensor gauge bosons are thick wave lines. The Lorentz indices  $\alpha\dot{\alpha}$  and momentum  $k$  belong to the first tensor gauge boson, the  $\gamma\dot{\gamma}$  and momentum  $q$  belong to the second tensor gauge boson, and Lorentz index  $\beta$  and momentum  $p$  belong to the vector gauge boson.

and the corresponding free field equations for the tensor gauge field  $A_{\mu\lambda_1\lambda_2}$  in four-dimensional space-time describe the *propagation of helicity-three and one*  $h = \pm 3, \pm 1, \pm 1$  *massless charged gauge bosons* in agreement with the spectrum (3.13). There are no propagating negative norm states. The comparison of these equations with the Schwinger-Fronsdal equations [30, 31, 32, 33] can be found in [45].

Considering the free field equation for the general rank- $(s+1)$  tensor gauge field one can find that the quadratic part of the Lagrangian has the following form [29]:

$$\mathcal{L}_{s+1} + \mathcal{L}'_{s+1} |_{quadratic} = \frac{1}{2} A_{\alpha\lambda_1\dots\lambda_s}^a \mathcal{H}^{\alpha\lambda_1\dots\lambda_s \gamma\lambda_{s+1}\dots\lambda_{2s}} A_{\gamma\lambda_{s+1}\dots\lambda_{2s}}^a \quad (4.19)$$

and is invariant with respect to the group of gauge transformations

$$\delta A_{\alpha\lambda_1\dots\lambda_s}^a = \partial_\alpha \xi_{\lambda_1\dots\lambda_s}^a, \quad \tilde{\delta} A_{\alpha\lambda_1\dots\lambda_s}^a = \partial_{\lambda_1} \zeta_{\lambda_2\dots\lambda_s\alpha}^a + \dots + \partial_{\lambda_s} \zeta_{\lambda_1\dots\lambda_{s-1}\alpha}^a, \quad (4.20)$$

which should fulfil the following constraints:

$$\begin{aligned} \partial_\rho \zeta_{\rho\lambda_1\dots\lambda_{s-1}}^a - \frac{1}{s-2} (\partial_{\lambda_1} \zeta_{\lambda_2\dots\lambda_{s-1}\rho\rho}^a + \dots + \partial_{\lambda_{s-1}} \zeta_{\lambda_1\dots\lambda_{s-2}\rho\rho}^a) &= 0, \\ \partial_{\lambda_1} \zeta_{\lambda_2\dots\lambda_{s-1}\rho\rho}^a - \partial_{\lambda_2} \zeta_{\lambda_1\dots\lambda_{s-1}\rho\rho}^a &= 0. \end{aligned} \quad (4.21)$$

In momentum representation the kinetic operator has the following general form:

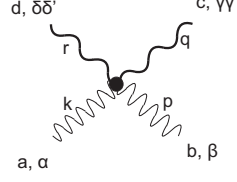


Figure 2: The quartic vertex with two vector gauge bosons and two tensor gauge bosons - the VVTT vertex -  $\mathcal{V}_{\alpha\beta\gamma\dot{\gamma}\delta\dot{\delta}}^{abcd}(k, p, q, r)$  in non-Abelian tensor gauge field theory [11]. Vector gauge bosons are conventionally drawn as thin wave lines, tensor gauge bosons are thick wave lines. The Lorentz indices  $\gamma\dot{\gamma}$  and momentum  $q$  belong to the first tensor gauge boson,  $\delta\dot{\delta}$  and momentum  $r$  belong to the second tensor gauge boson, the index  $\alpha$  and momentum  $k$  belong to the first vector gauge boson and Lorentz index  $\beta$  and momentum  $p$  belong to the second vector gauge boson.

$$\begin{aligned}
\mathcal{H}_{\alpha\lambda_1\ldots\lambda_s\ \gamma\lambda_{s+1}\ldots\lambda_{2s}} = & + \frac{1}{s!} \left( \sum_p \eta_{\lambda_{i_1}\lambda_{i_2}} \cdots \eta_{\lambda_{i_{2s-1}}\lambda_{i_{2s}}} \right) (-k^2 \eta_{\alpha\gamma} + k_\alpha k_\gamma) \\
& + \frac{1}{(s+1)!} \left( \sum_P \eta_{\alpha\lambda_{i_1}} \eta_{\lambda_{i_2}\lambda_{i_3}} \cdots \eta_{\lambda_{i_{2s-2}}\lambda_{i_{2s-1}}} \eta_{\gamma\lambda_{i_{2s}}} \right) k^2 \\
& - \frac{1}{(s+1)!} \left( \sum_P \eta_{\rho\lambda_{i_1}} \eta_{\lambda_{i_2}\lambda_{i_3}} \cdots \eta_{\lambda_{i_{2s-2}}\lambda_{i_{2s-1}}} \eta_{\gamma\lambda_{i_{2s}}} \right) k_\alpha k_\rho \quad (4.22) \\
& - \frac{1}{(s+1)!} \left( \sum_P \eta_{\rho\lambda_{i_1}} \eta_{\lambda_{i_2}\lambda_{i_3}} \cdots \eta_{\lambda_{i_{2s-2}}\lambda_{i_{2s-1}}} \eta_{\alpha\lambda_{i_{2s}}} \right) k_\rho k_\gamma \\
& + \frac{1}{(s+1)!} \eta_{\alpha\gamma} \left( \sum_P \eta_{\rho\lambda_{i_1}} \eta_{\lambda_{i_2}\lambda_{i_3}} \cdots \eta_{\lambda_{i_{2s-2}}\lambda_{i_{2s-1}}} \eta_{\sigma\lambda_{i_{2s}}} \right) k_\rho k_\sigma,
\end{aligned}$$

where the sum  $\sum_P$  runs over all non-equal permutations of  $\lambda_i$  's. The solution of the free field equation for the rank-(s+1) field [29]

$$\mathcal{H}^{\alpha\lambda_1\ldots\lambda_s\ \gamma\lambda_{s+1}\ldots\lambda_{2s}} A_{\gamma\lambda_{s+1}\ldots\lambda_{2s}} = 0. \quad (4.23)$$

describes the propagation of the helicities:

$$h = \pm(s+1), \quad \begin{matrix} \pm(s-1) \\ \pm(s-1) \end{matrix}, \quad \begin{matrix} \pm(s-3) \\ \pm(s-3) \end{matrix}, \quad \cdots \quad (4.24)$$

It is convenient to represent the spectrum (4.24) of tensor gauge bosons in the form which combines the helicity spectrum of all bosons. It is unbounded and has the following form

[50]:

$$\begin{aligned}
& \pm 1 \\
& \pm 2, \quad 0 \\
& \pm 3, \quad \pm 1, \quad \pm 1 \\
& \pm 4, \quad \pm 2, \quad \pm 2, \quad 0 \\
& \pm 5, \quad \pm 3, \quad \pm 3, \quad \pm 1, \quad \pm 1 \\
& \pm 6, \quad \pm 4, \quad \pm 4, \quad \pm 2, \quad \pm 2, \quad 0 \\
& \dots\dots\dots
\end{aligned} \tag{4.25}$$

In summary, we defined the composite gauge field (2.3) which takes a value in the transversal representation (3.7), (3.10), (3.11) of the extended Poincaré algebra  $L_G(\mathcal{P})$ . We constructed the invariant Lagrangian (4.16), (4.8), (4.12) which contains infinity many tensor gauge fields (2.1) and found their helicity content (4.24), (4.25).

The theory has unexpected symmetry with respect to the duality transformation of the gauge fields [89, 90]. The complementary gauge transformation  $\tilde{\delta}$  is defined as:

$$\begin{aligned}
\tilde{\delta} A_\mu^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \eta^b, \\
\tilde{\delta} A_{\mu\lambda_1}^a &= (\delta^{ab} \partial_{\lambda_1} + g f^{acb} A_{\lambda_1}^c) \eta_\mu^b + g f^{acb} A_{\mu\lambda_1}^c \eta^b, \\
\tilde{\delta} A_{\mu\lambda_1\lambda_2}^a &= (\delta^{ab} \partial_{\lambda_1} + g f^{acb} A_{\lambda_1}^c) \eta_{\mu\lambda_2}^b + (\delta^{ab} \partial_{\lambda_2} + g f^{acb} A_{\lambda_2}^c) \eta_{\mu\lambda_1}^b + \\
&\quad + g f^{acb} (A_{\mu\lambda_1}^c \eta_{\lambda_2}^b + A_{\mu\lambda_2}^c \eta_{\lambda_1}^b + A_{\lambda_1\lambda_2}^c \eta_\mu^b + A_{\lambda_2\lambda_1}^c \eta_\mu^b + A_{\mu\lambda_1\lambda_2}^c \eta^b), \\
&\dots\dots\dots
\end{aligned} \tag{4.26}$$

The transformations  $\delta$  in (4.5) and  $\tilde{\delta}$  in (4.26) do not coincide and are *complementary* to each other in the following sense: in  $\delta$  the derivatives of the gauge parameters  $\{\xi\}$  are over the first index  $\mu$ , while in  $\tilde{\delta}$  the derivatives of the gauge parameters  $\{\eta\}$  are over the rest of the totally symmetric indices  $\lambda_1 \dots \lambda_s$ . One can construct the new field strength tensors  $\tilde{G}_{\mu\nu, \lambda_1 \dots \lambda_s}^a$  which are transforming homogeneously with respect to the  $\tilde{\delta}$  transformations and then to construct the corresponding gauge invariant Lagrangian  $\tilde{L}(A)$  [89, 90]. The relation between these two Lagrangians was found in the form of duality transformation



$$\begin{aligned}\tilde{A}_{\mu\lambda_1} &= A_{\lambda_1\mu}, \\ \tilde{A}_{\mu\lambda_1\lambda_2} &= \frac{1}{2}(A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}) - \frac{1}{2}A_{\mu\lambda_1\lambda_2}, \\ \tilde{A}_{\mu\lambda_1\lambda_2\lambda_3} &= \frac{1}{3}(A_{\lambda_1\mu\lambda_2\lambda_3} + A_{\lambda_2\mu\lambda_1\lambda_3} + A_{\lambda_3\mu\lambda_1\lambda_2}) - \frac{2}{3}A_{\mu\lambda_1\lambda_2\lambda_3}, \\ &\dots\dots\dots\end{aligned}\tag{4.27}$$

The Lagrangian (4.16) defines not only a free propagation of tensor gauge bosons, but also their interactions. The interaction diagrams for the lower-rank bosons are presented on Fig.1-2. The high-rank bosons also interact through the triple and quartic interaction vertices. It is therefore important to calculate and study the scattering amplitudes, the quantum loop corrections and their high energy behaviour. By using the diagram technique it is possible to calculate the scattering amplitude, but the difficulties lie in the evaluation and contraction of high-rank tensors structures appearing in the diagram approach. In the next section we shall use alternative approach based on spinor representation of amplitudes developed recently in [57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74].

A scattering amplitude for the massless particles of momenta  $p_i$  and polarisation tensors  $\varepsilon_i$  ( $i = 1, \dots, n$ ), which are described by irreducible massless representations of the Poincaré group, can be represented in the following form:

It is more convenient to represent the momenta  $p_i$  and polarisation tensors  $\varepsilon_i$  in terms of spinors. In that case the scattering amplitude  $M_n$  can be considered as a function of spinors  $\lambda_i, \tilde{\lambda}_i$  and helicities  $h_i$  [57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74]:

The advantage of the spinor representation is that introducing a complex deformation of the particles momenta one can derive a general form for the three-particle interaction vertices [46, 47, 69, 75, 56]:

16

The dimensionality of the three-point vertex  $M_3(1^{h_1}, 2^{h_2}, 3^{h_3})$  is

$$[mass]^{D=\pm(h_1+h_2+h_3)}.$$

In the generalised Yang-Mills theory [9, 10, 11, 50], which we described in the previous sections, all interaction vertices between high-spin particles have *dimensionless coupling constants*, which means that the helicities of the interacting particles in the vertex are constrained by the relation

$$D = \pm(h_1 + h_2 + h_3) = 1.$$

Therefore the interaction vertex between massless tensor-bosons, the TTT-vertex, has the following general form [75, 56]:

$$\begin{aligned} M_3 &= g f^{abc} \langle 1, 2 \rangle^{-2h_1-2h_2-1} \langle 2, 3 \rangle^{2h_1+1} \langle 3, 1 \rangle^{2h_2+1}, \quad h_3 = -1 - h_1 - h_2, \\ M_3 &= g f^{abc} [1, 2]^{2h_1+2h_2-1} [2, 3]^{-2h_1+1} [3, 1]^{-2h_2+1}, \quad h_3 = 1 - h_1 - h_2, \end{aligned} \quad (5.2)$$

where  $f^{abc}$  are the structure constants of the internal gauge group G. In particular, considering the interaction between a boson of helicity  $h_1 = \pm 1$  and a tensor-boson of helicity  $h_2 = \pm s$ , the VTT-vertex, one can find from (5.2) that

$$h_3 = \pm|s-2|, \pm s, \pm|s+2| \quad (5.3)$$

and the corresponding vector-tensor-tensor interaction vertices VTT have the following form:

$$\begin{aligned} M_3^{a_1 a_2 a_3}(1^{-s}, 2^{-1}, 3^{+s}) &= g f^{a_1 a_2 a_3} \frac{\langle 1, 2 \rangle^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \langle 3, 1 \rangle} \left( \frac{\langle 1, 2 \rangle}{\langle 2, 3 \rangle} \right)^{2s-2}, \\ M_3^{a_1 a_2 a_3}(1^{-s}, 2^{+1}, 3^{s-2}) &= g f^{a_1 a_2 a_3} \frac{\langle 1, 3 \rangle^4}{\langle 1, 2 \rangle \langle 2, 3 \rangle \langle 3, 1 \rangle} \left( \frac{\langle 1, 2 \rangle}{\langle 2, 3 \rangle} \right)^{2s-2}. \end{aligned} \quad (5.4)$$

These are the vertices which reduce to the standard triple YM vertex when  $s = 1$ . Using these vertices one can compute the scattering amplitudes of vector and tensor bosons. The colour-ordered scattering amplitudes involving two tensor-bosons of helicities  $h = \pm s$ , one negative helicity vector-boson and  $(n-3)$  vector-bosons of positive helicity were found in [75]:

$$\hat{M}_n(1^+, \dots i^-, \dots k^{+s}, \dots j^{-s}, \dots n^+) = ig^{n-2} (2\pi)^4 \delta^{(4)}(P^{ab}) \frac{\langle ij \rangle^4}{\prod_{l=1}^n \langle ll+1 \rangle} \left( \frac{\langle ij \rangle}{\langle ik \rangle} \right)^{2s-2}, \quad (5.5)$$

where  $n$  is the total number of particles and the dots stand for any number of positive helicity vector-bosons,  $i$  is the position of the negative-helicity vector, while  $k$  and  $j$  are the positions of the tensors with helicities  $+s$  and  $-s$  respectively. The expression (5.5) reduces to the famous Parke-Taylor formula [62] when  $s = 1$ . In particular, the five-particle amplitude takes the following form:

$$\hat{M}_5(1^+, 2^-, 3^+, 4^{+s}, 5^{-s}) = ig^3(2\pi)^4 \delta^{(4)}(P^{ab}) \frac{\langle 25 \rangle^4}{\prod_{i=1}^5 \langle ii+1 \rangle} \left( \frac{\langle 25 \rangle}{\langle 24 \rangle} \right)^{2s-2}, \quad (5.6)$$

where  $P^{ab} = \sum_{m=1}^n \lambda_m^a \tilde{\lambda}_m^b$  is the total momentum. Notice that the scattering amplitudes (5.5) and (5.6) have large validity area: in the limit  $s \rightarrow 1/2$  they reduce to the tree level gluon scattering amplitudes into a quark pair and into a pair of scalars as  $s \rightarrow 0$ .

The scattering amplitudes (5.5) and (5.6) can be used to extract splitting amplitudes of vector and tensor bosons [76]. The collinear behaviour of the tree amplitudes has the following factorised form [61, 62, 63, 73, 74]:

$$M_n^{tree}(\dots, a^{\lambda_a}, b^{\lambda_b}, \dots) \xrightarrow{a \parallel b} \sum_{\lambda=\pm 1} Split_{-\lambda}^{tree}(a^{\lambda_a}, b^{\lambda_b}) \times M_{n-1}^{tree}(\dots, P^\lambda, \dots), \quad (5.7)$$

where  $Split_{-\lambda}^{tree}(a^{\lambda_a}, b^{\lambda_b})$  denotes the splitting amplitude and the intermediate state  $P$  has momentum  $k_P = k_a + k_b$  and helicity  $\lambda$ . Considering the amplitude (5.6) in the limit when the particles 4 and 5 become collinear,  $k_4 \parallel k_5$ , that is,  $k_4 = zk_P$ ,  $k_5 = (1-z)k_P$ ,  $k_P^2 \rightarrow 0$  and  $z$  describes the longitudinal momentum sharing, one can deduce that the corresponding behaviour of spinors is  $\lambda_4 = \sqrt{z}\lambda_P$ ,  $\lambda_5 = \sqrt{1-z}\lambda_P$ , and that the amplitude (5.6) takes the following factorisation form [76]:

$$M_5(1^+, 2^-, 3^+, 4^{+s}, 5^{-s}) = A_4(1^+, 2^-, 3^+, P^-) \times Split_+(a^{+s}, b^{-s}), \quad (5.8)$$

where

$$Split_+(a^{+s}, b^{-s}) = \left( \frac{1-z}{z} \right)^{s-1} \frac{(1-z)^2}{\sqrt{z(1-z)}} \frac{1}{\langle a, b \rangle}. \quad (5.9)$$

In a similar way one can deduce that

$$Split_+(a^{-s}, b^{+s}) = \left( \frac{z}{1-z} \right)^{s-1} \frac{z^2}{\sqrt{z(1-z)}} \frac{1}{\langle a, b \rangle}. \quad (5.10)$$

Considering different collinear limits  $k_1 \parallel k_5$  and  $k_3 \parallel k_4$  one can get [76]

$$Split_{+s}(a^+, b^{-s}) = \frac{(1-z)^{s+1}}{\sqrt{z(1-z)}} \frac{1}{\langle a, b \rangle}, \quad Split_{+s}(a^{-s}, b^+) = \frac{z^{s+1}}{\sqrt{z(1-z)}} \frac{1}{\langle a, b \rangle} \quad (5.11)$$

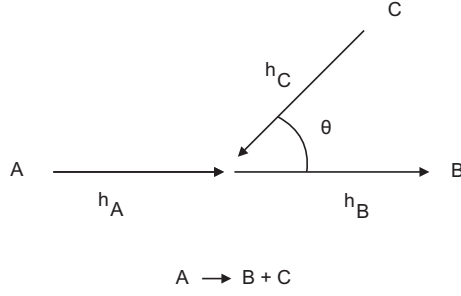


Figure 3: The decay of a gluon of helicity  $h_A$  into the tensorgluons of helicities  $h_B$  and  $h_C$ . The arrows show the directions of the helicities. The corresponding splitting probability is defined as  $P_{BA}$ .

and

$$Split_{-s}(a^{+s}, b^+) = \frac{z^{-s+1}}{\sqrt{z(1-z)}} \frac{1}{\langle a, b \rangle}, \quad Split_{-s}(a^+, b^{+s}) = \frac{(1-z)^{-s+1}}{\sqrt{z(1-z)}} \frac{1}{\langle a, b \rangle}. \quad (5.12)$$

The set of splitting amplitudes (5.9)-(5.12)  $V \rightarrow TT$ ,  $T \rightarrow VT$  and  $T \rightarrow TV$  reduces to the full set of gluon splitting amplitudes [61, 62, 63, 73, 74] when  $s = 1$ .

Since the collinear limits of the scattering amplitudes are responsible for parton evolution [78] we can extract from the above expressions the Altarelli-Parisi splitting probabilities for tensor-bosons. Indeed, the residue of the collinear pole in the square (of the factorised amplitude (5.7)) gives Altarelli-Parisi splitting probability  $P(z)$ :

$$P(z) = C_2(G) \sum_{h_P, h_a, h_b} |Split_{-h_P}(a^{h_a}, b^{h_b})|^2 s_{ab}, \quad (5.13)$$

where  $s_{ab} = 2k_a \cdot k_b = \langle a, b \rangle [a, b]$ . The invariant operator  $C_2$  for the representation  $R$  is defined by the equation  $t^a t^a = C_2(R) 1$  and  $tr(t^a t^b) = T(R) \delta^{ab}$ . Substituting the splitting amplitudes (5.9)-(5.12) into (5.13) we are getting

$$\begin{aligned} P_{TV}(z) &= C_2(G) \left[ \frac{z^4}{z(1-z)} \left( \frac{z}{1-z} \right)^{2s-2} + \frac{(1-z)^4}{z(1-z)} \left( \frac{1-z}{z} \right)^{2s-2} \right], \\ P_{VT}(z) &= C_2(G) \left[ \frac{1}{z(1-z)} \left( \frac{1}{1-z} \right)^{2s-2} + \frac{(1-z)^4}{z(1-z)} (1-z)^{2s-2} \right], \\ P_{TT}(z) &= C_2(G) \left[ \frac{z^4}{z(1-z)} z^{2s-2} + \frac{1}{z(1-z)} \left( \frac{1}{z} \right)^{2s-2} \right]. \end{aligned} \quad (5.14)$$

The momentum conservation in the vertices clearly fulfils because these functions satisfy the relations

$$P_{TV}(z) = P_{TV}(1-z), \quad P_{VT}(z) = P_{TT}(1-z), \quad z < 1. \quad (5.15)$$

In the leading order the kernel  $P_{TV}(z)$  has a meaning of variation per unit transfer momentum of the probability density of finding a tensor-boson inside the vector-boson,  $P_{VT}(z)$  - of finding a vector inside the tensor and  $P_{TT}(z)$  - of finding a tensor inside the tensor. For completeness we shall present also quark and vector-boson kernels [79, 82, 78, 80, 81]:

$$\begin{aligned} P_{qq}(z) &= C_2(R) \frac{1+z^2}{1-z}, \\ P_{Vq}(z) &= C_2(R) \frac{1+(1-z)^2}{z}, \\ P_{qV}(z) &= T(R)[z^2 + (1-z)^2], \\ P_{VV}(z) &= C_2(G) \left[ \frac{1}{z(1-z)} + \frac{z^4}{z(1-z)} + \frac{(1-z)^4}{z(1-z)} \right], \end{aligned} \quad (5.16)$$

where  $C_2(G) = N$ ,  $C_2(R) = \frac{N^2-1}{2N}$ ,  $T(R) = \frac{1}{2}$  for the SU(N) groups.

*Having in hand the new set of splitting probabilities for tensor-bosons (5.14) we can consider a possible generalisation of quantum chromodynamics [55]. In so generalised theory in addition to the quarks and gluons there should be tensorgluons. We can hypothesise that a possible emission of tensorgluons by gluons, as it is shown on Fig.1,3 should produce a non-zero density of tensorgluons inside the proton in addition to the quark and gluon densities. Our next goal is to derive DGLAP equations [78, 80, 81, 82, 83, 84, 85, 79] which will take into account these new emission processes.*

## 6 Generalization of DGLAP Equation.

### Calculation of Callan-Simanzik Beta Function

In this section we shall consider a possibility that inside the proton and, more generally, inside hadrons there are additional partons - tensorgluons, which can carry a part of the proton momentum [54, 55, 56]. Tensorgluons have zero electric charge, like gluons, but have a larger spin. Inside the proton a nonzero density of the tensorgluons can be generated by the emission of tensorgluons by gluons [9, 10, 11, 50]. The last mechanism is typical for non-Abelian tensor gauge theories, in which there exists a gluon-tensor-tensor vertex of order  $g$  (see Fig.1-2) [9, 10, 11, 50]. Therefore a number of gluons changes not only because a quark may radiate a gluon or because a gluon may split into a quark-antiquark pair or into two gluons [86, 87, 78], but also because a gluon can split into two tensorgluons [9, 10, 11, 50, 75, 76]. The process of gluon splitting into tensorgluons

suggests that part of the proton momentum which was carried by neutral partons can be shared between vector and tensor gluons. Our aim is to calculate the scattering amplitudes and splitting function in QCD generalised in this way.

It is well known that the deep inelastic structure functions can be expressed in terms of quark distribution densities. If  $q^i(x)$  is the density of quarks of type  $i$  (summed over colours) inside a proton target with fraction  $x$  of the proton longitudinal momentum in the infinite momentum frame [77] then the scaling structure functions can be represented in the following form:

$$2F_1(x) = F_2(x)/x = \sum_i Q_i^2 [q^i(x) + \bar{q}^i(x)]. \quad (6.1)$$

The scaling behaviour of the structure functions is broken and the results can be formulated by assigning a well determined  $Q^2$  dependence to the parton densities. This can be achieved by introducing integro-differential equations which describe the  $Q^2$  dependence of quark  $q^i(x, t)$  and gluon densities  $G(x, t)$ , where  $t = \ln(Q^2/Q_0^2)$  [78, 80, 81, 82, 83, 84, 85, 79].

Let us see what will happen if one supposes that there are additional partons - tensor gluons - inside the proton. In accordance with our hypothesis there is an additional emission of tensor gluons in the proton, therefore one should introduce the corresponding density  $T(x, t)$  of tensor gluons (summed over colours) inside the proton in the  $P_\infty$  frame [77]. We can derive integro-differential equations that describe the  $Q^2$  dependence of parton densities in this general case [55]:

$$\begin{aligned} \frac{dq^i(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ \sum_{j=1}^{2n_f} q^j(y, t) P_{q^i q^j}\left(\frac{x}{y}\right) + G(y, t) P_{q^i G}\left(\frac{x}{y}\right) \right], \\ \frac{dG(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ \sum_{j=1}^{2n_f} q^j(y, t) P_{G q^j}\left(\frac{x}{y}\right) + G(y, t) P_{GG}\left(\frac{x}{y}\right) + T(y, t) P_{GT}\left(\frac{x}{y}\right) \right], \\ \frac{dT(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ G(y, t) P_{TG}\left(\frac{x}{y}\right) + T(y, t) P_{TT}\left(\frac{x}{y}\right) \right]. \end{aligned} \quad (6.2)$$

The  $\alpha(t)$  is the running coupling constant ( $\alpha = g^2/4\pi$ ). In the leading logarithmic approximation  $\alpha(t)$  is of the form

$$\frac{\alpha}{\alpha(t)} = 1 + b \alpha t, \quad (6.3)$$

where  $\alpha = \alpha(0)$  and  $b$  is the one-loop Callan-Simanzik coefficient, which, as we shall see below, receives an additional contribution from the tensorgluon loop. Here the indices  $i$  and  $j$  run over quarks and antiquarks of all flavors. The number of quarks of a given fraction of momentum changes when a quark loses momentum by radiating a gluon, or a gluon inside the proton may produce a quark-antiquark pair [78]. Similarly the number of gluons changes because a quark may radiate a gluon or because a gluon may split into a quark-antiquark pair or into two gluons or *into two tensorgluons*. This last possibility is realised, because, as we have seen, in non-Abelian tensor gauge theories there is a triple vertex VTT (5.4) of a gluon and two tensorgluons of order  $g$  [9, 10, 11, 50]. This interaction should be taken into consideration, and we added the term  $T(y, t) P_{GT}(\frac{x}{y})$  in the second equation (6.2). The density of tensorgluons  $T(x, t)$  changes when a gluon splits into two tensorgluons or when a tensorgluon radiates a gluon. This evolution is described by the last equation (6.2).

In order to guarantee that the total momentum of the proton, that is, of all partons is unchanged, one should impose the following constraint:

$$\frac{d}{dt} \int_0^1 dz z \left[ \sum_{i=1}^{2n_f} q^i(z, t) + G(z, t) + T(z, t) \right] = 0. \quad (6.4)$$

Using the evolution equations (6.2) one can express the derivatives of the densities in (6.4) in terms of kernels and to see that the following momentum sum rules should be fulfilled:

$$\begin{aligned} \int_0^1 dz z [P_{qq}(z) + P_{Gq}(z)] &= 0, \\ \int_0^1 dz z [2n_f P_{qG}(z) + P_{GG}(z) + P_{TG}(z)] &= 0, \\ \int_0^1 dz z [P_{GT}(z) + P_{TT}(z)] &= 0. \end{aligned} \quad (6.5)$$

Before analysing these momentum sum rules let us first inspect the behaviour of the gluon-tensorgluon kernels (5.14) at the end points  $z = 0, 1$ . As one can see, they are singular at the boundary values similarly to the case of the standard kernels (5.16). Though there is a difference here: the singularities are of higher order compared to the standard case [78]. Therefore one should define the regularisation procedure for the singular factors  $(1 - z)^{-2s+1}$  and  $z^{-2s+1}$  reinterpreting them as the distributions  $(1 - z)_+^{-2s+1}$  and  $z_+^{-2s+1}$ , similarly to the Altarelli-Parisi regularisation [78]. We shall define them in the following

way:

$$\begin{aligned}
\int_0^1 dz \frac{f(z)}{(1-z)_+^{2s-1}} &= \int_0^1 dz \frac{f(z) - \sum_{k=0}^{2s-2} \frac{(-1)^k}{k!} f^{(k)}(1)(1-z)^k}{(1-z)^{2s-1}}, \\
\int_0^1 dz \frac{f(z)}{z_+^{2s-1}} &= \int_0^1 dz \frac{f(z) - \sum_{k=0}^{2s-2} \frac{1}{k!} f^{(k)}(0)z^k}{z^{2s-1}}, \\
\int_0^1 dz \frac{f(z)}{z_+(1-z)_+} &= \int_0^1 dz \frac{f(z) - (1-z)f(0) - zf(1)}{z(1-z)},
\end{aligned} \tag{6.6}$$

where  $f(z)$  is any test function which is sufficiently regular at the end points and, as one can see, the defined subtraction guarantees the convergence of the integrals. Using the same arguments as in the standard case [78] we should add the delta function terms into the definition of the diagonal kernels so that they will completely determine the behaviour of  $P_{qq}(z)$ ,  $P_{GG}(z)$  and  $P_{TG}(z)$  functions. The first equation in the momentum sum rule (6.5) remains unchanged because there is no tensorgluon contribution into the quark evolution. The second equation in the momentum sum rule (6.5) will take the following form:

$$\begin{aligned}
&\int_0^1 dz z [2n_f P_{qG}(z) + P_{GG}(z) + P_{TG}(z) + b_G \delta(z-1)] = \\
&= \int_0^1 dz z [2n_f T(R)[z^2 + (1-z)^2] + C_2(G) \left[ \frac{1}{z(1-z)} + \frac{z^4}{z(1-z)} + \frac{(1-z)^4}{z(1-z)} \right] + \\
&\quad + C_2(G) \left[ \frac{z^4}{z(1-z)} \left( \frac{z}{1-z} \right)^{2s-2} + \frac{(1-z)^4}{z(1-z)} \left( \frac{1-z}{z} \right)^{2s-2} \right] + b_G = \\
&= \frac{2}{3} n_f T(R) - \frac{11}{6} C_2(G) - \frac{12s^2-1}{6} C_2(G) + b = 0.
\end{aligned} \tag{6.7}$$

From this result we can extract an additional contribution to the one-loop Callan-Symanzik beta function arising from the tensorgluon loop. Indeed, the first beta-function coefficient enters into this expression because the momentum sum rule (6.5) implicitly comprises unitarity, thus the one-loop effects [78]. In (6.7) we have three terms which come from gluon and quark loops:

$$b_1 = \frac{11}{6} C_2(G) - \frac{2n_f}{3} T(R), \tag{6.8}$$

and from the tensorboson loop of spin  $s$ :

$$b_T = \frac{12s^2-1}{6} C_2(G), \quad s = 1, 2, 3, 4, \dots \tag{6.9}$$

It is a very interesting result because at  $s=1$  we are rediscovering the asymptotic freedom result [86, 87, 88]. For larger spins the tensorgluon contribution into the Callan-Symanzik



beta function has the same signature as the standard gluons, which means that tensorgluons "accelerate" the asymptotic freedom (6.3) of the strong interaction coupling constant  $\alpha(t)$ . The contribution is increasing quadratically with the spin of the tensorgluons, that is, at large transfer momentum the strong coupling constant tends to zero faster compared to the standard case:

$$\alpha(t) = \frac{\alpha}{1 + b\alpha t}, \quad (6.10)$$

where

$$b = \frac{(12s^2 - 1)C_2(G) - 4n_f T(R)}{12\pi}, \quad s = 1, 2, \dots \quad (6.11)$$

Surprisingly, a similar result based on the parametrization of the charge renormalization taken in the form  $b = (-1)^{2s}(A + Bs^2)$  was conjectured by Curtright [92]. Here  $A$  represents an orbital contribution and  $Bs^2$  - the anomalous magnetic moment contribution [93, 94, 95]. The unknown coefficients  $A$  and  $B$  were found by comparing the suggested parametrisation with the known results for  $s = 0, 1/2$  and  $1$ .

It is also possible to consider a straitforward generalisation of the result obtained for the effective action in Yang-Mills theory long ago [93, 94, 95, 96] to the higher spin gauge bosons. With the spectrum of the tensorgluons in the external chromomagnetic field  $\lambda = (2n + 1 + 2s)gH + k_{\parallel}^2$  one can perform a summation of the modes and get an exact result for the one-loop effective action similarly to [93, 96]:

$$\epsilon = \frac{H^2}{2} + \frac{(gH)^2}{4\pi} b \left[ \ln \frac{gH}{\mu^2} - \frac{1}{2} \right], \quad (6.12)$$

where

$$b = -\frac{2C_2(G)}{\pi} \zeta\left(-1, \frac{2s+1}{2}\right) = \frac{12s^2 - 1}{12\pi} C_2(G), \quad (6.13)$$

and  $\zeta(-1, q) = -\frac{1}{2}(q^2 - q + \frac{1}{6})$  is the generalised zeta function<sup>2</sup>. Because the coefficient in front of the logarithm defines the beta function [93, 94], one can see that (6.13) is in agreement with the result (6.9).

It is also natural to ask what will happen if one takes into consideration the contribution of tensorgluons of all spins into the beta function<sup>3</sup>. One can suggest two scenarios. In the first one the high spin gluons, let us say, of  $s \geq 3$ , will get large mass and therefore they can be ignored at a given energy scale. In the second case, when all of them remain

---

<sup>2</sup>The generalised zeta function is defined as  $\zeta(p, q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} dt t^{-1+p} \frac{e^{-qt}}{1-e^{-t}}$ .

<sup>3</sup>I would like to thank John Iliopoulos and Constantin Bachas for raising this question.

massless, then one can suggest the Riemann zeta function regularisation, similar to the Brink-Nielsen regularisation [97]. The summation over the spectrum in (4.25) gives[56]:

$$\begin{aligned}
b_{11} &= C_2(G) \left[ \sum_{s=1}^{\infty} \frac{(12s^2 - 1)}{12\pi} + \sum_{s=0}^{\infty} \frac{(12s^2 - 1)}{12\pi} + \sum_{s=1}^{\infty} \frac{(12s^2 - 1)}{12\pi} + \sum_{s=0}^{\infty} \frac{(12s^2 - 1)}{12\pi} + \dots \right] \\
&= C_2(G) \left[ \frac{1}{\pi} \zeta(-2) - \frac{1}{12\pi} \zeta(0) - \frac{1}{12\pi} + \frac{1}{\pi} \zeta(-2) - \frac{1}{12\pi} \zeta(0) + \dots \right] \\
&= C_2(G) \left[ \frac{1}{24\pi} - \frac{1}{12\pi} + \frac{1}{24\pi} + \dots \right] = 0,
\end{aligned} \tag{6.14}$$

where  $\zeta(-2) = 0$ ,  $\zeta(0) = -1/2$ , leading to the theory which is *conformally invariant* at very high energies. The above summation requires explicit regularisation and further justification.

## 7 Unification of Coupling Constants of Standard Model

It is interesting to know how the contribution of tensorgluons changes the high energy behaviour of the coupling constants of the Standard Model [99, 100]. The coupling constants are evolving in accordance with the formulae

$$\frac{1}{\alpha_i(M)} = \frac{1}{\alpha_i(\mu)} + 2b_i \ln \frac{M}{\mu}, \quad i = 1, 2, 3, \tag{7.1}$$

where we shall consider only the contribution of the lower  $s = 2$  tensorbosons:

$$2b = \frac{58C_2(G) - 4n_f T(R)}{6\pi}. \tag{7.2}$$

For the  $SU(3)_c \times SU(2)_L \times U(1)$  group with its coupling constants  $\alpha_3, \alpha_2$  and  $\alpha_1$  and six quarks  $n_f = 6$  and  $SU(5)$  unification group we will get

$$2b_3 = \frac{1}{2\pi} 54, \quad 2b_2 = \frac{1}{2\pi} \frac{104}{3}, \quad 2b_1 = -\frac{1}{2\pi} 4,$$

so that solving the system of equations (7.1) one can get

$$\ln \frac{M}{\mu} = \frac{\pi}{58} \left( \frac{1}{\alpha_{el}(\mu)} - \frac{8}{3} \frac{1}{\alpha_s(\mu)} \right), \tag{7.3}$$

where  $\alpha_{el}(\mu)$  and  $\alpha_s(\mu)$  are the electromagnetic and strong coupling constants at scale  $\mu$ . If one takes  $\alpha_{el}(M_Z) = 1/128$  and  $\alpha_s(M_Z) = 1/10$  one can get that coupling constants have equal strength at energies of order

$$M \sim 4 \times 10^4 GeV = 40 TeV,$$

which is much smaller than the scale  $M \sim 10^{14} GeV$  in the absence of the tensorgluons contribution. The value of the weak angle [99, 100] remains intact :

$$\sin^2 \theta_W = \frac{1}{6} + \frac{5}{9} \frac{\alpha_{el}(M_Z)}{\alpha_s(M_Z)}, \quad (7.4)$$

as well as the coupling constant at the unification scale remains of the same order  $\bar{\alpha}(M) = 0,01$ .

## 8 Conclusion

In the present article we describe a possible extension the Yang-Mills gauge principle [1] which includes tensor gauge fields. In this extension of the Yang-Mills theory the vector gauge boson becomes a member of a bigger family of gauge bosons of arbitrary large integer spins.

The proposed extension of Yang-Mills theory is essentially based on the existence of the enlarged Poincaré algebra and on an appropriate transversal representations of that algebra. The invariant Lagrangian is expressed in terms of new higher-rank field strength tensors. The Lagrangian does not contain higher derivatives of tensor gauge fields and all interactions take place through three- and four-particle exchanges with a dimensionless coupling constant (see Fig.1-2).

We calculated the scattering amplitudes of non-Abelian tensor gauge bosons at tree level, as well as their one-loop contribution into the Callan-Symanzik beta function. This contribution is negative and corresponds to the asymptotically free theory. The proposed extension may lead to a natural inclusion of the standard theory of fundamental forces into a larger theory in which vector gauge bosons, leptons and quarks represent a low-spin subgroup.

In the line with the above development we considered a possible extension of QCD. In so extended QCD inside the proton and, more generally, inside hadrons there should be additional partons - tensorgluons, which can carry a part of the proton momentum. Among all parton distributions, the gluon density  $G(x, t)$  is one of the least constrained functions since it does not couple directly to the photon in deep-inelastic scattering measurements of the proton  $F_2$  structure function. Therefore it is only indirectly constrained by scaling violations and by the momentum sum rule which resulted in the fact that only

half of the proton momentum is carried by charged constituents - the quarks - and that the other part is ascribed to the neutral constituents.

As it was suggested, the process of gluon splitting leads to the emission of tensorgluons and therefore a part of the proton momentum which is carried by the neutral constituents can be shared between gluons and tensorgluons. The density of neutral partons in the proton is therefore given by the sum of two functions:  $G(x, t) + T(x, t)$ , where  $T(x, t)$  is the density of the tensorgluons. To disentangle these contributions and to decide which piece of the neutral partons is the contribution of gluons and which one is of the tensorgluons one should measure the helicities of the neutral components, which seems to be a difficult task.

The gluon density can be directly constrained by jet production [98]. In the suggested model the situation is such that the standard quarks cannot radiate tensorgluons (such a vertex is absent in the model [9, 10, 11, 50]), therefore only gluons are radiated by quarks. A radiated gluon then can split into a pair of tensorgluons without obscuring the structure of the observed three-jet final states. Thus it seems that there is no obvious contradiction with the existing experimental data. Our hypotheses may be wrong, but the uniqueness and simplicity of suggested extension seems to be the reasons for serious consideration.

This extension of QCD influences the unification scale at which the coupling constants of the Standard Model merge. In the last section we observed that the unification scale at which standard coupling constants are merging is shifted to lower energies telling us that it may be that a new physics is round the corner. Whether all these phenomena are consistent with experiment is an open question.

## 9 *Acknowledgement*

I would like to thank the organisers of the "Conference on 60 Years of Yang-Mills Gauge Field Theories" for their kind hospitality in the Institute of Advanced Studies of the Nanyang Technological University in Singapore and Prof. Kok-Khoo Phua and Prof. Yong Min Cho for invitation. This work was supported in part by the General Secretariat for Research and Technology of Greece and the European Regional Development Fund (NSRF 2007-15 ACTION, KRIPIS).

# References

- [1] C.N.Yang and R.L.Mills. *Conservation of Isotopic Spin and Isotopic Gauge Invariance*. Phys. Rev. **96** (1954) 191
- [2] S.S.Chern. *Topics in Defferential Geometry*, Ch. III "Theory of Connections"  
(The Institute for Advanced Study, Princeton, 1951)
- [3] H. Weyl, *Electron and Gravitation.*, Z. Phys. **56** (1929) 330
- [4] H. Weyl, *Space, Time, Matter*, (Dover, New York, 1952).
- [5] E. Cartan, *Sur les variétés á connexion affine et la théorie de la relativité généralisée*. Annales Sci. Ecole Norm. Sup. **40** (1923) 325; **41** (1924) 1; **42** (1925) 17
- [6] E. Cartan, *La méthode de repére mobile, la théorie des groupes continus, et les espaces généralisés*, (Hermann, Paris 1935).
- [7] R. Utiyama, *Invariant theoretical interpretation of interaction*, Phys. Rev. **101** (1956) 1597
- [8] T. W. B. Kibble, *Lorentz invariance and the gravitational field*, J. Math. Phys. **2** (1961) 212.
- [9] G. Savvidy, *Non-Abelian tensor gauge fields: Generalization of Yang-Mills theory*, Phys. Lett. B **625** (2005) 341
- [10] G. Savvidy, *Non-abelian tensor gauge fields. I*, Int. J. Mod. Phys. A **21** (2006) 4931;
- [11] G. Savvidy, *Non-abelian tensor gauge fields. II*, Int. J. Mod. Phys. A **21** (2006) 4959;
- [12] H.Yukawa, *Quantum Theory of Non-Local Fields. Part I. Free Fields*, Phys. Rev. **77** (1950) 219
- [13] M. Fierz, *Non-Local Fields*, Phys. Rev. **78** (1950) 184
- [14] E. Wigner, *Invariant Quantum Mechanical Equations of Motion*, in *Theoretical Physics ed. A.Salam* (International Atomic Energy, Vienna, 1963) p 59

- [15] G. K. Savvidy, *Conformal Invariant Tensionless Strings*, Phys. Lett. B **552** (2003) 72.
- [16] G. K. Savvidy, *Tensionless strings: Physical Fock space and higher spin fields*, Int. J. Mod. Phys. A **19**, (2004) 3171-3194.
- [17] G. Savvidy, *Tensionless strings, correspondence with  $SO(D,D)$  sigma model*, Phys. Lett. B **615** (2005) 285.
- [18] Yu. A. Golfand and E. P. Likhtman, *Extension of the Algebra of Poincare Group Generators and Violation of  $p$  Invariance*, JETP Lett. **13** (1971) 323
- [19] A. Salam and J. A. Strathdee, *Feynman Rules for Superfields*, Nucl. Phys. B **86** (1975) 142.
- [20] S. R. Coleman and J. Mandula, *All possible symmetries of the  $S$  matrix*, Phys. Rev. **159** (1967) 1251.
- [21] R. Haag, J. T. Lopuszanski and M. Sohnius, *All Possible Generators Of Supersymmetries Of The  $S$  Matrix*, Nucl. Phys. B **88** (1975) 257.
- [22] E. Wigner. *On Unitary Representations of the Inhomogeneous Lorentz Group*. Ann. Math. **40** (1939) 149.
- [23] E. Majorana. *Teoria Relativistica di Particelle con Momento Intrinseco Arbitrario*, Nuovo Cimento **9** (1932) 335
- [24] P.A.M. Dirac. *Relativistic wave equations*, Proc. Roy. Soc. **A155** (1936) 447;  
*Unitary Representation of the Lorentz Group*, Proc. Roy. Soc. **A183** (1944) 284.
- [25] M. Fierz. *Über die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin*, Helv. Phys. Acta. **12** (1939) 3.
- [26] M. Fierz and W. Pauli. *On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field*, Proc. Roy. Soc. **A173** (1939) 211.
- [27] P. Minkowski. *Versuch einer konsistenten Theorie eines Spin-2 Mesons*, Helv. Phys. Acta. **32** (1966) 477

- [28] G. Savvidy, *Particle spectrum of Non-Abelian tensor gauge fields*, Mod. Phys. Lett. A **25** (2010) 1137 [arXiv:0909.3859 [hep-th]].
- [29] G. Savvidy, *Solution of free field equations in non-Abelian tensor gauge field theory*, Phys. Lett. B **682** (2009) 143.
- [30] J.Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, MA, 1970)
- [31] L. P. S. Singh and C. R. Hagen, *Lagrangian formulation for arbitrary spin. I. The boson case*, Phys. Rev. **D9** (1974) 898
- [32] C.Fronsdal, *Massless fields with integer spin*, Phys.Rev. **D18** (1978) 3624
- [33] S. Weinberg, *Feynman Rules For Any Spin*, Phys. Rev. **133** (1964) B1318.
- [34] V. L. Ginzburg and I. E. Tamm, *To the theory of spin*, ZETP **17** (1947) 227
- [35] P. Ramond, *Dual Theory for Free Fermions*, Phys. Rev. D **3** (1971) 2415.
- [36] L. Brink, *The Emergence of anticommuting coordinates and the Dirac-Ramond-Kostant operators*, Comment. Phys. Math. Soc. Sci. Fenn. **166** (2004) 11 [hep-th/0403211].
- [37] V. L. Ginzburg and V. I. Manko, *Relativistic Wave Equations with Inner Degrees of Freedom and Partons*, Sov. J. Part. Nucl. **7** (1976) 1
- [38] Y. Nambu, *Relativistic groups and infinite-component fields*, In \*N. Svartholm, Elementary Particle Theory. Proceedings Of The Nobel Symposium Held 1968 At Lerum, Sweden\*, Stockholm 1968, 105-117
- [39] E.S. Fradkin and M.A. Vasiliev, Dokl. Acad. Nauk. 29, 1100 (1986); Ann. of Phys. 177, 63 (1987).
- [40] M.A.Vasiliev et. al. *Nonlinear Higher Spin Theories in Various Dimensions* hep-th/0503128
- [41] F. A. Berends, G. J. H Burgers and H. Van Dam, *On the Theoretical problems in Constructing Interactions Involving Higher-Spin Massless Particles*, Nucl. Phys. B **260** (1985) 295.

- [42] A. Sagnotti, E. Sezgin and P. Sundell, *On higher spins with a strong  $Sp(2,R)$  condition*, arXiv:hep-th/0501156.
- [43] R. R. Metsaev, *Cubic interaction vertices of massive and massless higher spin fields*, Nucl. Phys. B **759** (2006) 147
- [44] R. Manvelyan, K. Mkrtchyan and W. Ruhl, *General trilinear interaction for arbitrary even higher spin gauge fields*, Nucl. Phys. B **836** (2010) 204
- [45] S. Guttenberg and G. Savvidy, *Schwinger-Fronsdal Theory of Abelian Tensor Gauge Fields*, SIGMA **4** (2008) 061 [arXiv:0804.0522 [hep-th]].
- [46] A. K. Bengtsson, I. Bengtsson and L. Brink, *Cubic Interaction Terms For Arbitrary Spin*, Nucl. Phys. B **227** (1983) 31.
- [47] A. K. Bengtsson, I. Bengtsson and L. Brink, *Cubic Interaction Terms For Arbitrarily Extended Supermultiplets*, Nucl. Phys. B **227** (1983) 41.
- [48] E. Gabrielli, *Extended pure Yang-Mills gauge theories with scalar and tensor gauge fields*, Phys. Lett. B **258** (1991) 151. doi:10.1016/0370-2693(91)91223-I
- [49] C. Castro, *On generalized Yang-Mills theories and extensions of the standard model in Clifford (tensorial) spaces*, Annals Phys. **321** (2006) 813. doi:10.1016/j.aop.2005.11.008
- [50] G. Savvidy, *Extension of the Poincaré Group and Non-Abelian Tensor Gauge Fields*, Int. J. Mod. Phys. A **25** (2010) 5765 [arXiv:1006.3005 [hep-th]].
- [51] G. Savvidy, *Non-Abelian Tensor Gauge Fields*, Proc. Steklov Inst. Math. **272** (2011) 201 [arXiv:1004.4456 [hep-th]].
- [52] I. Antoniadis, L. Brink and G. Savvidy, *Extensions of the Poincare group*, J. Math. Phys. **52** (2011) 072303 [arXiv:1103.2456 [hep-th]].
- [53] G. Savvidy, *Invariant scalar product on extended Poincar algebra*, J. Phys. A **47** (2014) 5, 055204 [arXiv:1308.2695 [hep-th]].
- [54] G. Savvidy, *Asymptotic freedom of non-Abelian tensor gauge fields*, Phys. Lett. B **732** (2014) 150.



- [55] G. Savvidy, *Proton structure and tensor gluons*, J. Phys. A **47** (2014) 35, 355401 [arXiv:1310.0856 [hep-th]].
- [56] G. Savvidy, *Tensor gluons and proton structure*, Theor. Math. Phys. **182** (2015) 1, 114 [Teor. Mat. Fiz. **182** (2014) 1, 140] [arXiv:1406.5334 [hep-ph]].
- [57] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans and T. T. Wu, *Single Bremsstrahlung Processes In Gauge Theories*, Phys. Lett. B **103** (1981) 124.
- [58] R. Kleiss and W. J. Stirling, *Spinor Techniques For Calculating  $P$  Anti- $P \rightarrow W^{+-}/Z^0 + Jets$* , Nucl. Phys. B **262** (1985) 235.
- [59] Z. Xu, D. H. Zhang and L. Chang, *Helicity Amplitudes for Multiple Bremsstrahlung in Massless Nonabelian Gauge Theories*, Nucl. Phys. B **291** (1987) 392.
- [60] J. F. Gunion and Z. Kunszt, *Improved Analytic Techniques For Tree Graph Calculations And The  $G G Q$  Anti- $Q$  Lepton Anti-Lepton Subprocess*, Phys. Lett. B **161** (1985) 333.
- [61] L. J. Dixon, *Calculating scattering amplitudes efficiently*, arXiv:hep-ph/9601359.
- [62] S. J. Parke and T. R. Taylor, *An Amplitude for  $n$  Gluon Scattering*, Phys. Rev. Lett. **56** (1986) 2459.
- [63] F. A. Berends and W. T. Giele, *Recursive Calculations for Processes with  $n$  Gluons*, Nucl. Phys. B **306** (1988) 759.
- [64] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, Commun. Math. Phys. **252** (2004) 189 [arXiv:hep-th/0312171].
- [65] F. Cachazo, P. Svrcek and E. Witten, *Gauge theory amplitudes in twistor space and holomorphic anomaly*, JHEP **0410** (2004) 077 [hep-th/0409245].
- [66] F. Cachazo, *Holomorphic anomaly of unitarity cuts and one-loop gauge theory amplitudes*, hep-th/0410077.
- [67] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, Nucl. Phys. B **715** (2005) 499 [arXiv:hep-th/0412308].

- [68] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, Phys. Rev. Lett. **94** (2005) 181602 [arXiv:hep-th/0501052].
- [69] P. Benincasa and F. Cachazo, *Consistency Conditions on the S-Matrix of Massless Particles*, arXiv:0705.4305 [hep-th].
- [70] F. Cachazo, P. Svrcek and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, JHEP **0409** (2004) 006 [arXiv:hep-th/0403047].
- [71] G. Georgiou, E. W. N. Glover and V. V. Khoze, *Non-MHV Tree Amplitudes in Gauge Theory*, JHEP **0407** (2004) 048 [arXiv:hep-th/0407027].
- [72] N. Arkani-Hamed and J. Kaplan, *On Tree Amplitudes in Gauge Theory and Gravity*, JHEP **0804** (2008) 076 [arXiv:0801.2385 [hep-th]].
- [73] F. A. Berends and W. T. Giele, *Multiple Soft Gluon Radiation in Parton Processes*, Nucl. Phys. B **313** (1989) 595.
- [74] M. L. Mangano and S. J. Parke, *Quark - Gluon Amplitudes in the Dual Expansion*, Nucl. Phys. B **299** (1988) 673.
- [75] G. Georgiou and G. Savvidy, *Production of non-Abelian tensor gauge bosons. Tree amplitudes and BCFW recursion relation*, Int. J. Mod. Phys. A **26** (2011) 2537 [arXiv:1007.3756 [hep-th]].
- [76] I. Antoniadis and G. Savvidy, *Conformal invariance of tensor boson tree amplitudes*, Mod. Phys. Lett. A **27** (2012) 1250103 [arXiv:1107.4997 [hep-th]].
- [77] J. D. Bjorken and E. A. Paschos, *Inelastic Electron Proton and gamma Proton Scattering, and the Structure of the Nucleon*, Phys. Rev. **185** (1969) 1975.
- [78] G. Altarelli and G. Parisi, *Asymptotic Freedom in Parton Language*, Nucl. Phys. B **126** (1977) 298.
- [79] Y. L. Dokshitzer, *Calculation of the Structure Functions for Deep Inelastic Scattering and  $e^+ e^-$  Annihilation by Perturbation Theory in Quantum Chromodynamics.*, Sov. Phys. JETP **46** (1977) 641 [Zh. Eksp. Teor. Fiz. **73** (1977) 1216].

- [80] V. N. Gribov and L. N. Lipatov, *Deep inelastic  $e p$  scattering in perturbation theory*, Sov. J. Nucl. Phys. **15** (1972) 438 [Yad. Fiz. **15** (1972) 781].
- [81] V. N. Gribov and L. N. Lipatov,  *$e^+ e^-$  pair annihilation and deep inelastic  $e p$  scattering in perturbation theory*, Sov. J. Nucl. Phys. **15** (1972) 675 [Yad. Fiz. **15** (1972) 1218].
- [82] L. N. Lipatov, *The parton model and perturbation theory*, Sov. J. Nucl. Phys. **20** (1975) 94 [Yad. Fiz. **20** (1974) 181].
- [83] V. S. Fadin, E. A. Kuraev and L. N. Lipatov, *On the Pomeron singularity in Asymptotically Free Theories*, Phys. Lett. B **60** (1975) 50.
- [84] E. A. Kuraev, L. N. Lipatov and V. S. Fadin, *The Pomeron singularity in Nonabelian Gauge Theories*, Sov. Phys. JETP **45** (1977) 199 [Zh. Eksp. Teor. Fiz. **72** (1977) 377].
- [85] I. I. Balitsky and L. N. Lipatov, *The Pomeron singularity in Quantum Chromodynamics*, Sov. J. Nucl. Phys. **28** (1978) 822 [Yad. Fiz. **28** (1978) 1597].
- [86] D. J. Gross and F. Wilczek, *Asymptotically Free Gauge Theories. 1*, Phys. Rev. D **8** (1973) 3633.
- [87] D. J. Gross and F. Wilczek, *Asymptotically Free Gauge Theories. 2.*, Phys. Rev. D **9** (1974) 980.
- [88] H. D. Politzer, *Reliable Perturbative Results For Strong Interactions?*, Phys. Rev. Lett. **30**, 1346 (1973).
- [89] J. K. Barrett and G. Savvidy, *A dual lagrangian for non-Abelian tensor gauge fields*, Phys. Lett. B **652** (2007) 141
- [90] S. Guttenberg and G. Savvidy, *Duality transformation of non-Abelian tensor gauge fields*, Mod. Phys. Lett. A **23** (2008) 999 [arXiv:0801.2459 [hep-th]].
- [91] J. Fang and C. Fronsdal, *Deformation of gauge groups. Gravitation*. J. Math. Phys. **20** (1979) 2264

- [92] T. L. Curtright, *Charge Renormalization And High Spin Fields*, Phys. Lett. B **102** (1981) 17.
- [93] G. K. Savvidy, *Infrared Instability of the Vacuum State of Gauge Theories and Asymptotic Freedom*, Phys. Lett. B **71** (1977) 133.
- [94] S. G. Matinyan and G. K. Savvidy, *Vacuum Polarization Induced by the Intense Gauge Field*, Nucl. Phys. B **134** (1978) 539.
- [95] I. A. Batalin, S. G. Matinyan and G. K. Savvidy, *Vacuum Polarization by a Source-Free Gauge Field*, Sov. J. Nucl. Phys. **26** (1977) 214 [Yad. Fiz. **26** (1977) 407].
- [96] D. Kay, *Supersymmetric Savvidy State For  $Su(2)$ ,  $Su(3)$  And  $Su(4)$* , Phys. Rev. D **28** (1983) 1562.
- [97] L. Brink and H. B. Nielsen, *A physical interpretation of the jacobi imaginary transformation and the critical dimension in dual models*, Phys. Lett. B **43** (1973) 319.
- [98] J. R. Ellis, M. K. Gaillard and G. G. Ross, *Search for Gluons in  $e^+ e^-$  Annihilation*, Nucl. Phys. B **111** (1976) 253
- [99] H. Georgi and S. L. Glashow, *Unity of All Elementary Particle Forces*, Phys. Rev. Lett. **32** (1974) 438.
- [100] H. Georgi, H. R. Quinn and S. Weinberg, *Hierarchy of Interactions in Unified Gauge Theories*, Phys. Rev. Lett. **33** (1974) 451.